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THE EFFECT OF SLIGHT ANISOTROPY ON THE PROPERTIES OF SHOCK WAVES IN A COMPRESSIBLE ELASTIC MEDIUM†

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The effect of slight anisotropy on the behaviour of shock waves in a compressible elastic medium is investigated. Particular attention is paid to the properties of shock waves which are not close to plane-polarized (the properties of plane-polarized shock waves only change slightly). Some results are obtained for an arbitrary form of anisotropy when the behaviour of shock waves is known in media which differ from those considered in having no anisotropy. The effect of slight anisotropy on the behaviour of shock waves in an incompressible elastic medium has been considered previously in [1].

1. FORMULATION OF THE PROBLEM

We shall consider the one-dimensional motions of an elastic medium under the assumption that the dependence of the internal energy Φ per unit of initial volume on the derivatives of the components of the displacement vector w_i with respect to the Lagrangian coordinate x has the form [2]

$$\Phi(u_i,S)=F(u_1^2+u_2^2,u_3,S)+gp(u_i,S),\quad d\Phi=\Phi_k du_k+\rho T dS$$
$$\Phi_k=\partial\Phi/\partial u_k,\quad u_i=\partial w_i/\partial x\quad i,k=1,2,3 \tag{1.1}$$

where w_i are the components of the displacement vector in the Cartesian system of coordinates x_1, x_2, $x_3 = x$, g is a parameter which characterizes the anisotropy of the properties of the system in the plane of a wave (the wave anisotropy parameter) which is henceforth assumed to be small, F and p are certain functions of their arguments, ρ is the initial density of the medium which corresponds to $u_3 = 0$ and T is the temperature. The second equation is a consequence of the first and second laws of thermodynamics.

Relations which express the laws of conservation of momentum and energy, as well as the condition of the continuity of the displacements w_i in the shock wave which propagates at a velocity $dx/dt = W$, can be written in the form [3]

$$\rho W[\upsilon_k]+[\Phi_k]=0,\quad W[u_k]+[\upsilon_k]=0$$
$$W[\Phi+\rho\upsilon^2/2]+[u_i\Phi_i]=0,\quad \upsilon_k=\partial w_k/\partial t \tag{1.2}$$

Square brackets denote a jump in a quantity: $[F] = F - F^-$; quantities with a minus sign correspond to the state before the jump and those without a minus sign correspond to the state after the jump. It follows from (1.1) and (1.2) that

$$[F_k]-\rho W^2[u_k]=-g[p_k],\quad [F]-(F_k+F_k^-)[u_k]/2=-g[p]+g(p_k+p_k^-)[u_k]/2$$
$$F_k=\partial F/\partial u_k,\quad p_k=\partial p/\partial u_k \tag{1.3}$$

We shall call the set of states u_i of the medium across the shock wave which satisfy relations (1.2) for specified u_i^-, s^- the shock adiabatic curve (SAC). In the case of a general position, the SAC is a curve in the space u_i, at each point of which W and S can be found.

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2. SOME PROPERTIES OF SHOCK WAVES WHEN THERE IS NO WAVE ANISOTROPY

Shock waves have been studied previously [3–8] in the case of wave isotropy ($g = 0$). Here, we shall recall some results which refer to this case.

If $g = 0$, then it follows from the construction of the arguments of the function F that the SAC has the following form. Three branches of the SAC pass through the initial point A: two of them lie in a plane passing through the initial point A and the axis u_3 which we shall call the plane of the initial state while the third is a circle L_A lying in the plane $u_3 = u_3^-$ with its centre at the point $u_1 = 0, u_2 = 0$. Shock waves, states in which u_i lies in the plane of the initial state, will be said to be plane-polarized.

The circle L_A describes a set of states across a "rotational shock wave" with an initial state A. Such a shock wave can also be considered [3–8] as the limiting case of its own form of a time-invariant Riemann wave. The entropy in it does not change and the velocity of propagation of the rotational shock wave is constant: $W =$ const in L_A and is identical to the characteristic velocity on both sides of the shock. Using this, it is not difficult to obtain $W^2 = F_{22}^-$ if $u_2^- = 0$ or $W^2 = F_{22}$ if $u_2 = 0$ ($F_{22} = \partial^2 F/\partial u_2^2$).

In the case of plane polarized shock waves it is possible, by choosing, for example, a system of coordinates such that $u_2^- = 0, u_2 = 0$ to obtain the equations for the "plane" part of the SAC which lies in the plane of the initial state. In this case, the second equation (1.3), which corresponds to the projection on the x_2-axis is automatically satisfied and can be discarded.

If the system moves from the initial point A along one of the plane branches of the SAC, then a point B can be found on it at which the velocity of the discontinuity W_{AB} is identical to the velocity of the rotational shock wave corresponding to the point B. Then a circle L_B describing a set of states across the rotational shock wave from state B will also belong to the SAC. In this case, the shock wave AB corresponding to the jump from the point A and point B and the rotational shock wave when their velocities are identical can be considered as a single shock wave. Shock waves with final states u_i lying in the circles L_A and L_B are not plane polarized if $u_i \neq u_i(B)$.

The velocity of a jump from point A to an arbitrary point of the circle L_B is also identical to the velocity of the rotational shock wave from point A.

Actually, it is also possible to reach the point of the circle L_B being considered using a rotational shock wave from point A and a plane polarized shock wave to the point under consideration in L_B. The latter shock wave has a velocity equal to W_{AB}. It can be obtained by satisfying the conservation laws that a rotational shock wave with an initial point A has the same velocity. Hence, the velocity of the shock wave from A to any point in the circle L_B is identical to the velocity of the rotational waves in front of and behind this shock wave.

Note that the entropy is constant over the whole of the circle L_B and, moreover, jumps from point A to points of L_B are physically permissible if $S_B - S_A \geqslant 0$. A branch of the plant part of the SAC also passes through the second point of intersection of the circle L_B with the initial plane passing through A and the u_3-axis since, otherwise, this point would be an isolated point which satisfies the equations for the plane part of the SAC which does not correspond to the case of a general position.

3. THE EFFECT OF ANISOTROPY ON A SHOCK ADIABATIC CURVE

In the case of a small value of $g \neq 0$ the plane part of a SAC as well as the quantities W and S undergo small changes of the order of g and, moreover, the ends of the segments with evolutionary behaviour (the Jouguet points) and segments where $[S] \geqslant 0$ may also be displaced by an amount of the order of g.

We are particularly interested in the influence of a small value of g on those parts of the SAC which, when $g = 0$, pass into the circles L_A or L_B since the corresponding shock waves, when $g = 0$, are found on the edges of evolutionary behaviour (the velocity of the shock wave is identical to the velocity of the rotational waves on the two sides of the shock wave), and, moreover, the points of the circle L_A satisfy the condition $[S] = 0$. All of the subsequent discussion will be concerned with the study of just such shock waves which, when $g = 0$, correspond to the points of the circles L_A or L_B.

The small change associated with g can affect the evolutionary behaviours of the shock wave as well as on the sign of $[S]$ in the case of the circle passing through point A.

For small g, we linearize Eqs (1.3) with respect to g taking account of (1.1) without perturbing the initial state and assuming perturbations of the quantities δu_k, δS, δW^2 of the order of g. We rotate the axes u_1 and u_2 around the u_3-axis and thereby introduce the new variables u_1', u_2', $u_3' = u_3$ in such a way that $u_2' = 0$ (here, $u_2'^- \neq 0$ in the general case). Then, using the notation $F_S = \partial F/\partial S$, $F_{iS} = \partial^2 F/\partial u_i' \partial S$, $F_{ij} = \partial^2 F/\partial u_i' \partial u_j'$ taking account of the equalities $F_{12} = 0$, $F_{23} = 0$, $F_{2S} = 0$, $F_S = \rho T$, $W^2 = F_{22}$ and assuming, to be specific, that $\delta u_2' = 0$, we obtain

$$\begin{aligned}
&(F_{11} - W^2)\delta u_1' + F_{13}\delta u_3' + F_{1S}\delta S - [u_1']\delta W^2 = -g[p_1'], \quad -[u_2']\delta W^2 = -g[p_2'] \\
&F_{13}\delta u_1' + (F_{33} - W^2)\delta u_3' + F_{3S}\delta S - [u_3']\delta W^2 = -g[p_3'] \\
&([F_1] - F_{1k}[u_k'])\delta u_3' + ([F_3] - F_{3k}[u_k'])\delta u_3' + (2\rho T - F_{Sk}[u_k'])\delta S = \\
&= g\{-2[p] + (p_k' + p_k'^-)[u_k']\}, \quad p_i' = \partial p / \partial u_i'
\end{aligned} \tag{3.1}$$

It is assumed that the coefficients and the right-hand sides in this system of equations are known from the zeroth approximation. System (3.1) enables us to write out the solution explicitly which expresses $\delta u_1'$, $\delta u_3'$, δS and δW^2. The formula

$$\delta W^2 = g[p_2'] / [u_2'] \tag{3.2}$$

has a particularly simple form which follows from the second equality of (3.1). The quantities $\delta u_1'$ and $\delta u_3'$ characterize the distortion of the circle in the space u_1', u_2', u_3' which arises from the effect of the anisotropy.

While not attempting to provide a full investigation of this distortion, we note that, if the quantity $[p_2']$ does not tend to zero when $[u_2'] \to 0$ (that is, when the shock wave approximates to a plane-polarized shock wave) then, according to the second equation of (3.1), the quantity δW^2 tends to infinity, that is, it becomes greater than g in its order of magnitude. The latter is untrue in the neighbourhood of the initial point A where the ratio $[p_2']/[u_2']$ has a finite limit, but is the case of the general situation for other points of intersection of the circles with the plane part of the SAC. In all the remaining equations of (3.1), apart from the second, the right-hand sides can be neglected when $\delta W^2 \to \infty$ and the effect of anisotropy only shows up through the quantity δW^2 which is determined from (3.2). If such an approximation is adopted, then the first, third and fourth equations of (3.1) will give the change in the position of a point on the SAC which is the same as that when $g = 0$ on the plane part of the SAC with the same value of δW^2.

If the point B does not correspond to the extremum of W^2 when $g = 0$, then, when $g \neq 0$, the approach of a point, representing a state across the shock wave, on the spatial part of the SAC to the plane of the initial state is accompanied by a motion along the plane part of the SAC (corresponding to $g = 0$), the direction of which is determined by the sign of δW^2 and is different when this plane is approached from different sides if $[p_2'] \neq 0$. Hence, the intersections of the circle and the plane part of the SAC existing when $g = 0$ which occur at the point B and at points which are symmetric to the points A and B about the u_3-axis, are disconnected when $g \neq 0$ and, for small g, in the neighbourhood of these points of the SAC resembles a hyperbola with branches whose geometry is determined by the sign of $[p_2']$ and the direction of the change in W^2 along the plane part of the SAC when $g = 0$.

We recall that equalities (3.1) and (3.2) are written in the system of coordinates u_1', u_2', u_3' in which $u_2 = 0$. In another system of coordinates, which differs from that used above by a rotation about the u_3-axis, the components of the vectors with subscripts 1 and 2 undergo an orthogonal transformation.

If the condition $u_2^- = 0$ is adopted for such a system of coordinates as u_1, u_2, u_3, then equality (3.2) takes the form

$$\begin{aligned}
&\delta W^2 = g\frac{[p_1]\cos\vartheta + [p_2]\sin\vartheta}{R\sin\vartheta} \\
&R^2 = (u_1^-)^2 + (u_2^-)^2, \quad \vartheta = \operatorname{arctg}(u_2 / u_1)
\end{aligned} \tag{3.3}$$

4. THE EFFECT OF ANISOTROPY ON THE ENTROPY CHANGE IN A SHOCK WAVE

If we multiply the first equation of (3.1) by $[u_1']/2$ and the third equation by $[u_3']/2$ and add them to the fourth equation, we obtain

$$T\delta S = g\{-[p] + p_1'^-[u_1'] + p_3'^-[u_3']\} + \delta W^2([u_1']^2 + [u_3']^2 / 2 \tag{4.1}$$

For the part of the SAC which, when $g = 0$, transforms into a circle which does not pass through the initial point A, the quantity δS, calculated using this equality, only gives a small correction to the entropy change $[S]$ in the shock wave and therefore cannot affect the sign of this change.

In the case of a shock wave which, when $g = 0$, corresponds to a rotational wave from the point $A([u_3'] = [u_3] = 0)$ with a rotation of the vector u_1, u_2 through an angle ϑ in an anticlockwise direction around the u_3-axis, we obtain

$$\rho T\delta S = g\{-[p]+(p_1^- \cos\vartheta + p_2^- \sin\vartheta)(1-\cos\vartheta)R\} + \delta W^2 R^2 (1-\cos\vartheta)^2/2 \tag{4.2}$$

This equality is written in the same system of coordinates associated with the initial state as equality (3.3). The inequality $\delta S \geqslant 0$ separates out the physically admissible shock waves. For a specified function $p(u_1, S)$ which has to be taken in (4.2) in the neighbourhood of L_A, that is, when $S = S^-$, this inequality can be easily investigated and the segments of the SAC with $\delta S \geqslant 0$ can be found.

In order to obtain additional evidence concerning the values of the entropy at points of the SAC a general relation, which is not associated with the smallness of g, is also useful

$$\frac{T}{W}\frac{dS}{dW} = \sum_{i=1}^{3}[u_i]^2 \tag{4.3}$$

(the derivative is taken on the SAC with respect to the velocity of the shock wave W). In order to derive this relation, equalities (1.2) are differentiated with respect to W while taking account of the fact that u_i^- are constant. One then adds the last of the resulting equalities to the first three multiplied by $-\upsilon_i$, and the following three multiplied by $-\Phi_i$. A relation, similar to (4.3), has been obtained previously in [6]. In the case of small g for parts of the SAC which, as $g \to 0$, pass into the circle, the factor T/W can be assumed to be constant.

5. THE EFFECT OF ANISOTROPY ON THE EVOLUTION OF SHOCK WAVES

In formulating the conditions for the evolutionary behaviour of a shock wave when $g \neq 0$ we shall assume that, in the initial and final states, the characteristic velocities differ from one another by amounts which do not vanish when $g = 0$. The relations in the shock wave are given by equations (1.2) or (1.3). Then, in the case of small g, the conditions for evolutionary behaviour for the part of the SAC close to a circle can be formulated as conditions which are imposed on the sign of the quantity $\delta W^2 - \delta c_\vartheta^2$ with respect to the two sides of the shock wave and, by c_ϑ, we mean the characteristic velocity of a wave which becomes rotational when $g = 0$. We shall assume that, in front of and behind the shock wave, the characteristic velocities are numbered in the ascending order of their magnitudes so that c_ϑ may denote c_1, c_2 and c_3 depending on the relations between the characteristic velocities. The conditions for evolutionary behaviour are then written in the form

$$(\delta W^2 - \delta c_\vartheta^2)^- \geqslant 0, \quad (\delta W^2 - \delta c_\vartheta^2)^+ \leqslant 0 \tag{5.1}$$

$$(\delta W^2 - \delta c_\vartheta^2)^-(\delta W^2 - \delta c_\vartheta^2)^+ \geqslant 0 \tag{5.2}$$

$$(\delta W^2 - \delta c_\vartheta^2)^- \leqslant 0, \quad (\delta W^2 - \delta c_\vartheta^2)^+ \geqslant 0 \tag{5.3}$$

The system of two equations (5.1) gives the conditions for evolutionary behaviour if c_ϑ has the same number on different sides of the shock wave. Inequality (5.2) refers to the case when the number behind the shock wave is one less than the number in front of the shock wave and, finally, system (5.3) refers to the case when $c_\vartheta^- = c_3^-$, $c_\vartheta^+ = c_1^+$. If the number c_ϑ behind the shock wave is greater than the corresponding number in front of it, the shock wave being considered is non-evolutionary.

By considering an infinitely small quasirotational shock wave at the point A, we obtain $(\delta c^2)^- = (dp_2/du_2)_A$. Here, the minus superscript indicates that the derivative is taken at the initial point along the corresponding branch of the SAC. This expression as well as the equalities (3.2) or (3.3) enable us to determine the sign of the quantity $(\delta W^2 - \delta c_\vartheta^2)^-$ at each point of the part of the SAC being considered.

In order to complete the treatment of inequalities (5.1)–(5.3), we note that the quantity $(\delta W - \delta c_\vartheta)^+$ vanishes and changes sign on the SAC at points at which W reaches an extremum [6] (Jouguet points). On the quasirotational part, where $W = \delta W + \text{const}$, the sign of $(\delta W - \delta c_\vartheta)^+$ is determined by the sign of the derivative of δW, taken along the SAC which corresponds to $g = 0$, close to the point B. To determine the sign of $(\delta W - \delta c_\vartheta)^+$, we use the following consideration which is based on equality (4.3). Close to the velocity maxima and minima, there are close pairs of points which correspond to the same value of W (which is close to the extremal value). These points may represent the initial and final states for a shock wave of small amplitude which corresponds to a velocity W. Moreover, of these two points, the initial state corresponds to the smaller value of the entropy S and the final state to the larger value of S. In the initial state $\delta W - \delta c_\vartheta > 0$ and, in the final state, $\delta W - \delta c_\vartheta < 0$ [9]. By making use of the form of the right-hand side in (4.3), we can conclude that the sign of $\delta W - \delta c_\vartheta$ changes on passing through the minimum of W (which coincides with the minimum in S) from minus to plus if one moves along the SAC in the direction of increasing $[u_1]^2 + [u_2]^2 + [u_3]^2$. In the neighbourhood of the maximum of W (or S), there is an opposite change in the sign of the difference $\delta W - \delta c_\vartheta$.

Hence, the signs of the quantities occurring in relations (5.1)–(5.3) can be readily determined, which enables one to make a judgement concerning the evolutionary behaviour of the shock waves being studied. Apart from the conditions for evolutionary behaviour, in the case of real systems it is necessary simultaneously to satisfy the conditions that there is no decrease in the entropy. Equality (4.1) or (4.2) serves to check this point.

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